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f(x,y) fonksiyonunda x,y değişkenleri x=g(u,v), y=h(u,v) şeklinde u,v değişkenlerine bağlı ise. $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ türevleri aşağıdaki gibi hesaplanır.

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

ÖRNEK : $z = 2x^2 + xy - y^2 + 2x - 3y + 5,\ x = 2u - v,\ y = u + v$ olduğuna göre $\frac{\partial z}{\partial u}$ ve $\frac{\partial z}{\partial v}$ türevlerini hesaplayınız.

Çözüm : $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4x + y + 2) \cdot 2 + (x - 2y - 3) \cdot 1 = 9x + 1$

$= 9\,(2u - v) + 1 = 18u - 9v + 1$

ve

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (4x + y + 2) \cdot (-1) + (x - 2y - 3) \cdot 1$

$= -3x - 3y - 5 = -3\,(2u - v) - 3\,(u + v) - 5$

$= -9u - 5$

bulunur.

ÖRNEK : $z = \ln(x^2 + y^2),\ x = e^v \cos u,\ y = e^v \sin u$ olduğuna göre $\frac{\partial z}{\partial u}$ ve $\frac{\partial z}{\partial v}$ türevlerini hesaplayınız.

Çözüm : $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{2x}{x^2+y^2} \cdot (-e^v \sin u) + \frac{2y}{x^2+y^2} e^v \cos u$

$= \frac{2e^v \cos u\,(-e^v \sin u)}{e^{2v}(\cos^2 u + \sin^2 u)} + \frac{2e^v \sin u e^v \cos u}{e^{2v}(\cos^2 u + \sin^2 u)} = 0$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{2x}{x^2+y^2} e^v \cos u + \frac{2y}{x^2+y^2} e^v \sin u$

$$= \frac{2e^v \cos u e^v \cos u}{e^{2v}} + \frac{2e^v \sin u e^v \sin u}{e^{2v}}$$

$$= \frac{2e^{2v}(\cos^2 u + \sin^2 u)}{e^{2v}} = 2$$

ÖRNEK : $z = \frac{1}{x^2+y^2}$, $x = \cos 2t$, $y = \sin 2t$ olduğuna göre $\frac{dz}{dt}$ türevini hesaplayınız.

Çözüm : $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$= \frac{-2x}{(x^2+y^2)^2} \cdot (-2\sin 2t) + \frac{-2y}{(x^2+y^2)^2} \cdot 2\cos 2t$$

$$= \frac{4\cos 2t \sin 2t}{(\cos^2 2t + \sin^2 2t)^2} - \frac{4\sin 2t \cos 2t}{(\cos^2 2t + \sin^2 2t)^2} = 0$$

bulunur.

ORNEK : $u = f(x, y, z) = \ln(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, $z = t$ olduğuna göre $\frac{du}{dt}$ türevini hesaplayınız. Bu türevin $t = 0$ noktasındaki değerini bulunuz.

Çözüm : $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$

$$= \frac{2x}{x^2+y^2+z^2} \cdot (-\sin t) + \frac{2y}{x^2+y^2+z^2} \cdot \cos t + \frac{2z}{x^2+y^2+z^2} \cdot 1$$

$$= \frac{-2\cos t \sin t}{1+t^2} + \frac{2\sin t \cos t}{1+t^2} + \frac{2t}{1+t^2} = \frac{2t}{1+t^2}$$

olur. Bu türevin $t = 0$ için değeri

$$\left.\frac{du}{dt}\right|_{t=0} = \frac{2.0}{1+0} = 0$$

olarak bulunur.

ÖRNEK : $s = \ln(x^2 + y^2 + z^2)$, $x = u - v$, $y = u + v$, $z = 2\sqrt{uv}$ olduğuna göre $\frac{\partial s}{\partial u}$ ve $\frac{\partial s}{\partial v}$ türevlerini hesaplayınız.

Çözüm :

$$\frac{\partial s}{\partial u} = \frac{\partial s}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial s}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial s}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{2x}{x^2+y^2+z^2}\cdot 1 + \frac{2y}{x^2+y^2+z^2}\cdot 1 + \frac{2z}{x^2+y^2+z^2}\cdot\frac{v}{\sqrt{uv}}$$

$$= \frac{2(u-v)+2(u+v)}{(u-v)^2+(u+v)^2+4uv} + \frac{4\sqrt{uv}}{(u-v)^2+(u+v)^2+4uv}\cdot\frac{v}{\sqrt{uv}}$$

$$= \frac{2u-2v+2u+2v+4v}{u^2-2uv+v^2+u^2+2uv+v^2+4uv} = \frac{4(u+v)}{2\left(u^2+2uv+v^2\right)}$$

$$= \frac{2(u+v)}{(u+v)^2} = \frac{2}{u+v}$$

olur. Benzer şekilde

$$\frac{\partial s}{\partial v} = \frac{2}{u+v}$$

olacağı gösterilebilir.

4. $y = f(x)$, $x = g(u, v)$ ise

$$\frac{\partial y}{\partial u} = \frac{dy}{dx}\cdot\frac{\partial x}{\partial u} \quad \text{ve} \quad \frac{\partial y}{\partial v} = \frac{dy}{dx}\cdot\frac{\partial x}{\partial v}$$

olur.

ÖRNEK : $y = \arctan x$, $x = 2u + 3v$ olduğuna göre $3\frac{\partial y}{\partial u} - 2\frac{\partial y}{\partial v}$ ifadesini hesaplayınız.

Çözüm : $$\frac{\partial y}{\partial u} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial u} = \frac{1}{1+x^2} \cdot 2 = \frac{2}{1+(2u+3v)^2} ,$$

$$\frac{\partial y}{\partial v} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial v} = \frac{1}{1+x^2} \cdot 3 = \frac{3}{1+(2u+3v)^2} ,$$

olacağından

$$3\frac{\partial y}{\partial u} - 2\frac{\partial y}{\partial v} = \frac{6}{1+(2u+3v)^2} - \frac{6}{1+(2u+3v)^2} = 0$$

olur.

ÖRNEK : $z = f(x, y)$, $x = r\cos\varphi$, $y = r\sin\varphi$ olsun.

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \varphi}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

olduğunu gösteriniz.

Çözüm : $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \varphi} = \frac{\partial z}{\partial x} \cdot \cos\varphi + \frac{\partial z}{\partial y} \cdot \sin\varphi$$

ve

$$\frac{\partial z}{\partial \varphi} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \varphi} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \varphi} = \frac{\partial z}{\partial x}(-r\sin\varphi) + \frac{\partial z}{\partial y} r\cos\varphi$$

olacağından,

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \varphi}\right)^2=\left(\frac{\partial z}{\partial x}\cos\varphi+\frac{\partial z}{\partial y}\sin\varphi\right)^2+\frac{1}{r^2}\left(-\frac{\partial z}{\partial x}r\sin\varphi+\frac{\partial z}{\partial y}r\cos\varphi\right)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\varphi+2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\sin\varphi.\cos\varphi+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\varphi$$

$$+\frac{1}{r^2}\left[\left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\varphi-2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}r^2\sin\varphi.\cos\varphi+\left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\varphi\right]$$

$$=\left(\frac{\partial z}{\partial x}\right)^2\left(\cos^2\varphi+\sin^2\varphi\right)+\left(\frac{\partial z}{\partial y}\right)^2\left(\cos^2\varphi+\sin^2\varphi\right)=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

ÖRNEK : $z_{xx}-m^2\,z_{yy}=0$ denklemini

$$u = mx + y$$

$$v = -mx + y$$

eşitlikleriyle verilen yeni u ve v değişkenlerine göre yazınız. Burada $z = z\,(x, y)$ sürekli kısmi türevlere sahip bir fonksiyondur.

Çözüm :

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}\cdot m+\frac{\partial z}{\partial v}\cdot(-m)=m\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial}{\partial u}\left(m\frac{\partial z}{\partial u}-m\frac{\partial z}{\partial v}\right)\cdot m+\frac{\partial}{\partial v}\left(m\frac{\partial z}{\partial u}-m\frac{\partial z}{\partial v}\right).(-m)$$

$$=m^2\left(\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial u\partial v}\right)+m^2\left(-\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}\right)$$

$$=m^2\left(z_{uu}-2\,z_{uv}+z_{vv}\right)$$

olur

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot 1 = \frac{\partial z}{\partial v} + \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = z_{uu} + 2z_{uv} + z_{vv}$$

bulunur. Bu değerler verilen denklemde yerine yazılırsa

$$m^2 (z_{uu} - 2z_{uv} + z_{vv}) - m^2 (z_{uu} + 2z_{uv} + z_{vv}) = 0$$

bulunur. Gerekli sadeleştirmeler yapılırsa

$$z_{uv} = 0$$

denklemi elde edilir.